Meeting 14 Math 22

Section 11.7.1 Triple Integrals

Triple integral of $f(x,y,z)$ over $B=[a,b]\times[c,d]\times[e,f]$

$$\iiint_B f(x,y,z)\,dV = \int_e^f\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$

Example $f(x,y,z) = x-y+2z$ $\quad B=[-2,3]\times[1,4]\times[0,2]$

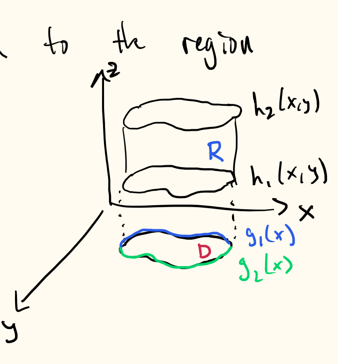
$$\iiint_B f\,dV = \int_{-2}^{3}\int_1^4\int_0^2 x-y+2z\;dz\,dy\,dx$$
$$= \int_{-2}^{3}\int_1^4 \left[xz-yz+z^2\right]_0^2 dy\,dx$$
$$= \int_{-2}^{3}\int_1^4 2x-2y+4\;dy\,dx$$
$$= \int_{-2}^{3}\left\{2xy-y^2+4y\right\}_1^4 dx$$
$$= \int_{-2}^{3} 8x-16+16-2x+1-4\;dx$$
$$= \int_{-2}^{3} 6x-3\,dx = 3x^2-3x\Big|_{-2}^{3}$$
$$= 27-9-12-6$$
$$= 0$$

You can evaluate a triple integral over a more general region R as long you can describe it as a solid bounded by the graphs of two continuous functions of two variables over a domain D bounded by two, functions of a single variable.

For instance,

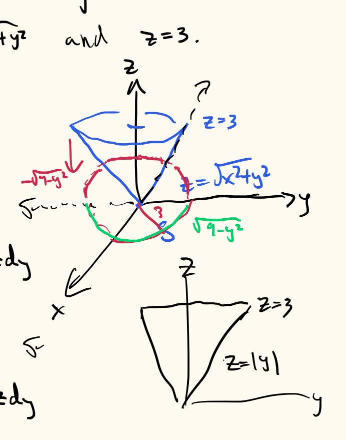
$$\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

would correspond to the region



Example Set up a triple integral over the solid bounded by $z=\sqrt{x^2+y^2}$ and $z=3$.

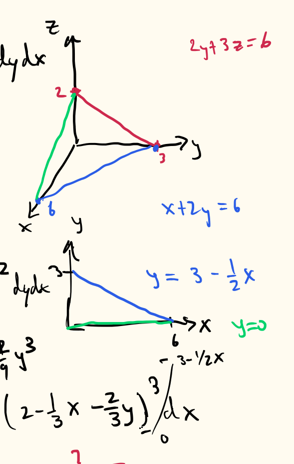
$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{3} f(x,y,z)\,dz\,dx\,dy$$



$$= \int_0^3\int_y^3\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x,y,z)\,dx\,dz\,dy$$
$$+ \int_{-3}^{0}\int_{-y}^{3}\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x,y,z)\,dx\,dz\,dy$$

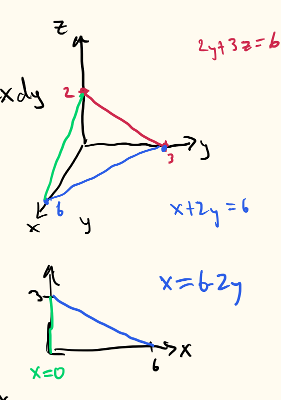
Example Solid bounded by $x+2y+3z=6$ in the first octant. Integrate the function $f(x,y,z)=x+y+z$.

$$\iiint x+y+z\,dV = \int_0^6\int_0^{3-\frac{1}{2}x}\int_0^{2-\frac{1}{3}x-\frac{2}{3}y} x+y+z\;dz\,dy\,dx$$
$$= \int_0^6\int_0^{3-\frac12 x}\left[xz+yz+\tfrac12 z^2\right]_0^{2-\frac13 x-\frac23 y} dy\,dx$$
$$= \int_0^6\int_0^{3-\frac12 x} x\left(2-\tfrac13 x-\tfrac23 y\right)+y\left(2-\tfrac13 x-\tfrac23 y\right) + \tfrac12\left(2-\tfrac13 x-\tfrac23 y\right)^2 dy\,dx$$
$$= \int_0^6 \Big[2xy-\tfrac13 x^2y-\tfrac13 xy^2+y^2-\tfrac16 xy^2-\tfrac29 y^3 + \tfrac16\cdot -\tfrac32\left(2-\tfrac13 x-\tfrac23 y\right)^3\Big]_0^{3-\frac12 x} dx$$
$$\overset{?}{=} \frac{15}{4}$$



Activity 11.7.3
- Complete w/ your group
- Class discussion.

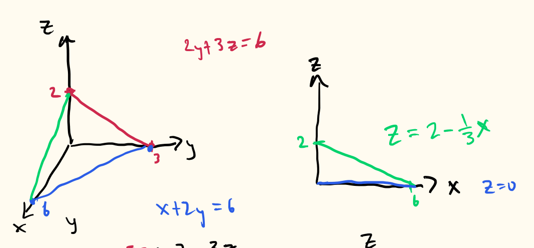
(b)
$$\iiint f(x,y,z)\,dV = \int_0^3\int_0^{6-2y}\int_0^{2-\frac13 x-\frac23 y} f\,dz\,dx\,dy$$



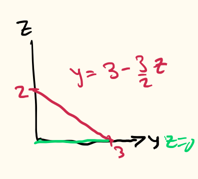
(c)
$$\iiint f\,dV = \int_0^6\int_0^{2-\frac13 x}\int_0^{3-\frac32 z-\frac12 x} f\,dy\,dz\,dx$$

$x+2y+3z=6$

$y = 3-\frac32 z-\frac12 x$

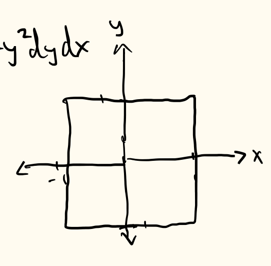


(d)
$$\iiint f\,dV = \int_0^2\int_0^{3-\frac32 z}\int_0^{6-2y-3z} f\,dx\,dy\,dz$$



Activity 11.7.4
- Complete w/ your group. Part (a) only.
- Class discussion.

(a)
$$\iiint_S 1\,dV = \int_{-1}^{1}\int_{-1}^{1}\int_0^{2-x^2-y^2} 1\,dz\,dy\,dx$$
$$= \int_{-1}^{1}\int_{-1}^{1} 2-x^2-y^2\,dy\,dx$$

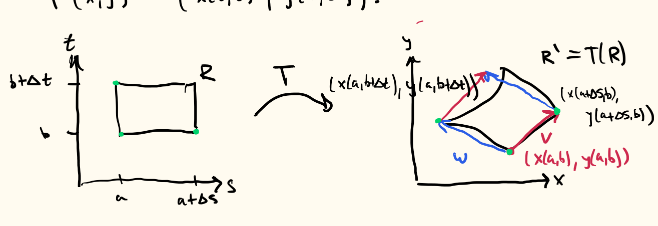


Section 11.9.2 Change of Variables for Double Integral

We want to evaluate a double integral $\iint_D f(x,y)\,dA$ by making a change of variables

$$x=x(s,t) \qquad y=y(s,t)$$

This defines a function $T:\mathbb{R}^2\to\mathbb{R}^2$ via $T(x,y) = (x(s,t),y(s,t))$.



In order to compute the integral, we need to determine dA in terms of s,t. We can approx. the area of R' using a linear approximation (a parallelogram).

We have

$$v = \left\langle \frac{x(a+\Delta s,b)-x(a,b)}{\Delta s}, \frac{y(a+\Delta s,b)-y(a,b)}{\Delta s}\right\rangle \Delta s$$

$$w = \left\langle \frac{x(a,b+\Delta t)-x(a,b)}{\Delta t}, \frac{y(a,b+\Delta t)-y(a,b)}{\Delta t}\right\rangle \Delta t$$

For small $\Delta s,\Delta t$, we have

$$v \approx \left\langle \frac{\partial x}{\partial s}(a,b), \frac{\partial y}{\partial s}(a,b)\right\rangle \Delta s$$
$$w \approx \left\langle \frac{\partial x}{\partial t}(a,b), \frac{\partial y}{\partial t}(a,b)\right\rangle \Delta t.$$

The area of T' is approximately

$$\Delta A = \left| \begin{vmatrix} \frac{\partial x}{\partial s}(a,b) & \frac{\partial y}{\partial s}(a,b) \\ \frac{\partial x}{\partial t}(a,b) & \frac{\partial y}{\partial t}(a,b)\end{vmatrix} \right| \Delta s\,\Delta t$$

the Jacobian. Notation: $\dfrac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t}\end{vmatrix}$

So, as $\Delta s,\Delta t\to 0$, we get

$$dA = \left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt.$$

Change of Variables formula Suppose $x(s,t), y(s,t)$ transforms a closed and bounded region R in st-coord. into a closed and bounded region R' in xy-coord. then

$$\iint_{R'} f(x,y)\,dA = \iint_R f(x(s,t),y(s,t))\left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt.$$

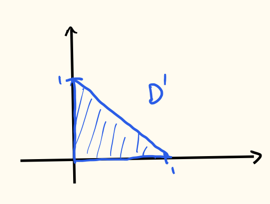
Activity 11.9.3
- Complete w/ your group
- Class discussion.

(a) $x=r\cos\theta \qquad y=r\sin\theta \qquad \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta}\end{vmatrix}$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta\end{vmatrix}$$
$$= r\cos^2\theta + r\sin^2\theta = r.$$

Activity 11.9.4
- Complete w/ your group
- Class discussion.

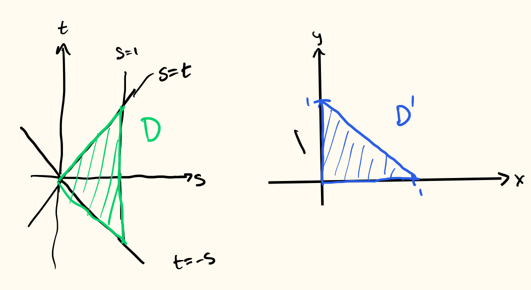
(a)



(c) $x = s-y = s-(x-t) = s-x+t$ $\quad \boxed{x=\frac{s+t}{2}}$

$y = s-x = s-(y+t) = s-y-t$ $\quad \boxed{y=\frac{s-t}{2}}$

(d)



(i) $\frac{s+t}{2}+\frac{s-t}{2}=1 \implies s=1$

(ii) $s+t=0$

(iii) $s-t=0$

(e)
$$\iint_{D'} \sqrt{x+y}\,(x-y)^2\,dA = \iint_D \sqrt{s}\,t^2\left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt$$

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = \left|\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2\end{vmatrix}\right| = \left|-\frac12\right| = \frac12$$

$$= \frac12\int_0^1\int_{-s}^{s}\sqrt{s}\,t^2\,dt\,ds$$
$$= \frac12\int_0^1 \frac13\sqrt{s}\,t^3\Big|_{-s}^{s} ds$$
$$= \frac16\int_0^1 \sqrt{s}(s^3+s^3)\,ds$$
$$= \frac13\int_0^1 s^{7/2}\,ds$$
$$= \frac13\cdot\frac29\, s^{9/2}\Big|_0^1$$
$$= \frac{2}{27}.$$

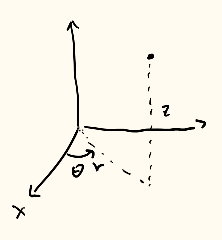
Section 11.9.3 Change of Variables for Triple Integrals

Suppose $x=x(s,t,u)$ $y=y(s,t,u)$ $z=z(s,t,u)$ transforms a closed and bounded solid S in stu-coord. into a closed and bounded solid S' in xyz-coordinates. Then

$$\iiint_{S'} f(x,y,z)\,dV = \iiint_S f(x(s,t,u),y(s,t,u),z(s,t,u))\left|\frac{\partial(x,y,z)}{\partial(s,t,u)}\right| ds\,dt\,du$$

where $\dfrac{\partial(x,y,z)}{\partial(s,t,u)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u}\end{vmatrix}$ is the Jacobian

Activity 11.9.5
- Complete w/ your group
- Class discussion.



$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1\end{vmatrix}$$
$$= r$$